Math Methods Unit 3 Test 3 2017
Anti - Differentiation

Resource Free **Time: 50 minutes** **Marks: / 46**

Only a formula sheet is allowed for this section. No calculator or notes allowed.

Question 1 **(12 marks)**

Evaluate each of the following, showing all working. Leave all answers with positive indices.

(a) $\int \frac{4}{t^2} dt$ (1 mark)

$$= \int 4t^{-2}\, dt$$

$$= \frac{4t^{-1}}{-1} + C$$

$$= \frac{-4}{t} + C$$

(b) $\int 3x(x^2-2)^3 dx$ (3 marks)

$$= \frac{3}{2}\int 2x(x^2-2)^3\, dx.$$

$$= \frac{3}{2}\left(\frac{(x^2-2)^4}{4}\right) + C$$

$$= \frac{3(x^2-2)^4}{8} + C.$$

(c) $\int (e^{-5x} + 2\pi x - \sqrt{x})\, dx$ (3 marks)

$$= \frac{e^{-5x}}{-5} + \frac{2\pi x^2}{2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= -\frac{1}{5e^{5x}} + \pi x^2 - \frac{2x^{3/2}}{3} + C.$$

(d) $\frac{d}{dx}\left(\int_{-3}^{x^2} \frac{\sqrt{2t-3}}{t+1}\,dt\right)$ (2 marks)

$$= \frac{\sqrt{2x^2-3}}{x^2+1} \times 2x.$$

If it is given that $f(x)$ is continuous everywhere and that $\int_4^{10} f(x)dx = -10$, find:

(e) $\int_1^3 f(3x+1)dx$ (3 marks)

$$= \frac{1}{3}\int_3^9 f(x+1)\,dx$$

$$= \frac{1}{3}\int_4^{10} f(x)\,dx$$

$$= \frac{1}{3}(-10) = -\frac{10}{3}.$$

Question 2 **(15 marks)**

Evaluate the following, showing full working.

(a) $\int_{-1}^{2}\left(x^2-1\right)dx$ (3 marks)

$$= \left[\frac{x^3}{3} - x\right]_{-1}^{2}$$

$$= \left(\frac{8}{3}-2\right) - \left(-\frac{1}{3}-(-1)\right)$$

$$= \frac{2}{3} - \frac{2}{3} = 0.$$

(b) $-3\int_{\pi}^{2\pi} \cos(3x)\,dx$ (3 marks)

$$= -\cancel{3}\left[\frac{\sin 3x}{\cancel{3}}\right]_{\pi}^{2\pi}$$

$$= -\left[\sin 6\pi - \sin 3\pi\right]$$

$$= -[0-0] = 0.$$

(c) $\int_{-1}^{3}\left(-e^{4x}+2\right)dx$ • (correct anti-diff) (3 marks)

$$= \left[-\frac{e^{4x}}{4} + 2x\right]_{-1}^{3}$$

• (correct substitution)

$$= \left(-\frac{e^{12}}{4} + 6\right) - \left(-\frac{e^{-4}}{4} - 2\right)$$

$$= -\frac{e^{12}}{4} + \frac{e^{-4}}{4} + 8.$$ • (final answer)

(d) $\frac{d}{dx}\int_{4}^{x^2}\frac{2}{3t^3-1}\,dt$ (3 marks)

$$= \frac{2}{3(x^2)^3-1} \times 2x$$

$$= \frac{4x}{3x^6-1}.$$

(e) $\int_{1}^{2}\frac{d}{dx}\left(\frac{x^3}{x^2+1}\right)dx$ (3 marks)

$$= \left[\frac{x^3}{x^2+1}\right]_{1}^{2}$$

$$= \frac{8}{5} - \frac{1}{2}$$

$$= \frac{16-5}{10}$$

$$= \frac{11}{10}$$

$$= 1.1.$$

Question 3 **(3 marks)**

The derivative of $f(x)$ is given by $f'(x) = 2e^{2x} + 3x^2$. Given that $f(1) = 4 + e^2$, find an expression for $f(x)$.

$$f(x) = \frac{\cancel{2} e^{2x}}{\cancel{2}} + \frac{\cancel{3} x^3}{\cancel{3}} + C$$

$$4 + e^2 = e^{2(1)} + (1)^3 + C$$

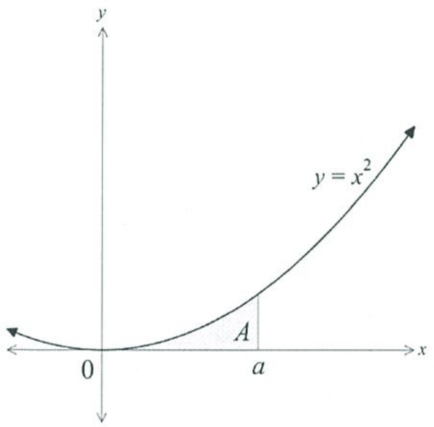
$$4 + \cancel{e^2} = \cancel{e^2} + 1 + C$$

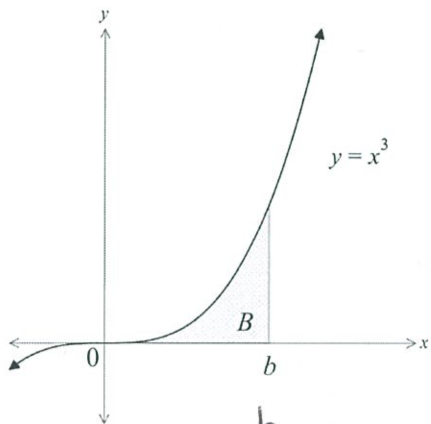
$$3 = C$$

$$\therefore f(x) = e^{2x} + x^3 + 3$$

Question 4 **(3 marks)**

The area labelled B is two times the area labelled A. Express b in terms of a.





$$\text{area } A = \int_0^a x^2 \, dx \qquad \text{area } B = \int_0^b x^3 \, dx$$

$$= \left[\frac{1}{3}x^3\right]_0^a \qquad = \left[\frac{1}{4}x^4\right]_0^b$$

$$A = \frac{1}{3}a^3 \qquad B = \frac{1}{4}b^4$$

but $B = 2A$

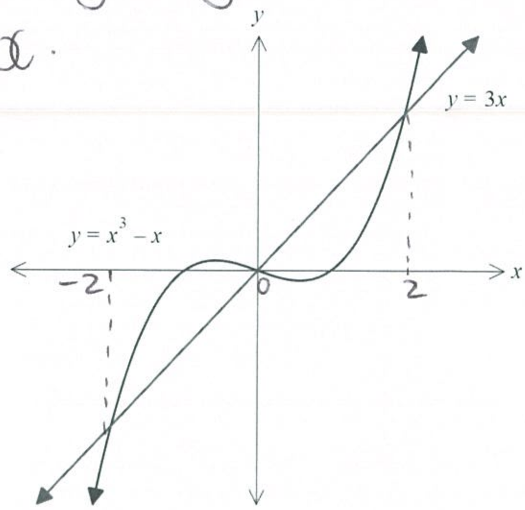
$$\therefore \frac{1}{4}b^4 = \frac{2a^3}{3} \qquad \therefore b^4 = \frac{8a^3}{3}$$

$$b = \sqrt[4]{\frac{8}{3}}\, a^{\frac{3}{4}}$$

Question 5 **(3 marks)**

Find the exact area bound by the two curves shown below.

(symmetry).



$$\text{Area} = 2\int_0^2 \left[3x - (x^3 - x)\right]\,dx$$

$$= 2\int_0^2 (4x - x^3)\,dx$$

$$= 2\left[2x^2 - \frac{x^4}{4}\right]_0^2$$

$$= 2(8-4)$$

$$= \underline{8 \text{ units}^2}.$$

for boundaries:

$$x^3 - x = 3x$$

$$x^3 - 4x = 0$$

$$x(x^2-4) = 0.$$

$$\underline{x = 0 \text{ or } x = \pm 2}.$$

Question 6 **(4 marks)**

Determine the function y given that $\frac{d^2y}{dx^2} = 3e^x + 2$ and $\frac{dy}{dx} = 5$ when $x = 0$ and $y = 3e^2 + 15$ when $x = 2$.

$$\frac{d^2y}{dx^2} = 3e^x + 2$$

$$\therefore \frac{dy}{dx} = 3e^x + 2x + C.$$

$$5 = 3e^0 + 2(0) + C$$

$$\therefore \underline{C = 2}.$$

$$\therefore \frac{dy}{dx} = 3e^x + 2x + 2.$$

$$\therefore y = 3e^x + x^2 + 2x + d.$$

$$\cancel{3e^2} + 15 = \cancel{3e^2} + 2^2 + 2(2) + d.$$

$$\therefore d = 7.$$

$$\therefore \underline{y = 3e^x + x^2 + 2x + 7}.$$

Question 7

(6 marks)

The gradient function of $f(x)$ is given by $f'(x) = ax^2 + b$. Determine the values of a and b if $f'(-2) = 28$, $f(0) = 1$ and $f(1) = 7$.

$f'(-2) = 4a+b$

$\therefore 28 = 4a+b.$ ✓

$f(x) = \frac{ax^3}{3} + bx + c.$ ✓

$1 = c$ ✓

$f(1)=7:$

$7 = \frac{a}{3} + b + 1.$ ✓

$\therefore 6 = \frac{a}{3} + b$

and $28 = 4a + b.$

$22 = \frac{11a}{3}$

$\frac{66}{11} = a$

$\therefore \underline{a=6}$ ✓

$28 = 4a+b.$

$28 = 24+b$

$\underline{b=4}$ ✓

METHODS YEAR 12 **Test 3 2017** **Name:** Sol key.
Anti-Differentiation

Resource Assumed **Time: 25 minutes** **Marks: / 25**

CAS calculator + A4 page 1 side of notes

Question 8 **(8 marks)**

Sam has invested $\$A$ in a fund which compounds her investment continuously at a rate of k % per annum.

The rate of change of her investment is given by $\frac{dV}{dt} = k(Ae^{kt})$ where V is the value of her investment in dollars and t is the time in years.

The net change in the value of her investment in the first 10 years is \$12 331 . 78.

The net change in the value of her investment in the next 10 years is \$22 469 . 97.

(a) Determine the values of A and k. (6 marks)

$$\int_0^{10} kAe^{kt}\,dt = 12\,331.78.$$

$$\int_{10}^{20} kAe^{kt}\,dt = 22469.97$$

$$\therefore [Ae^{kt}]_0^{10} = 12331.78 \quad \text{and} \quad [Ae^{kt}]_{10}^{20} = 22469.97$$

(1) $Ae^{10k} - A = 12331.78$ (2) $Ae^{20k} - Ae^{10k} = 22469.97$

Solve (1) and (2)

$\therefore$ $k = 0.06$

and

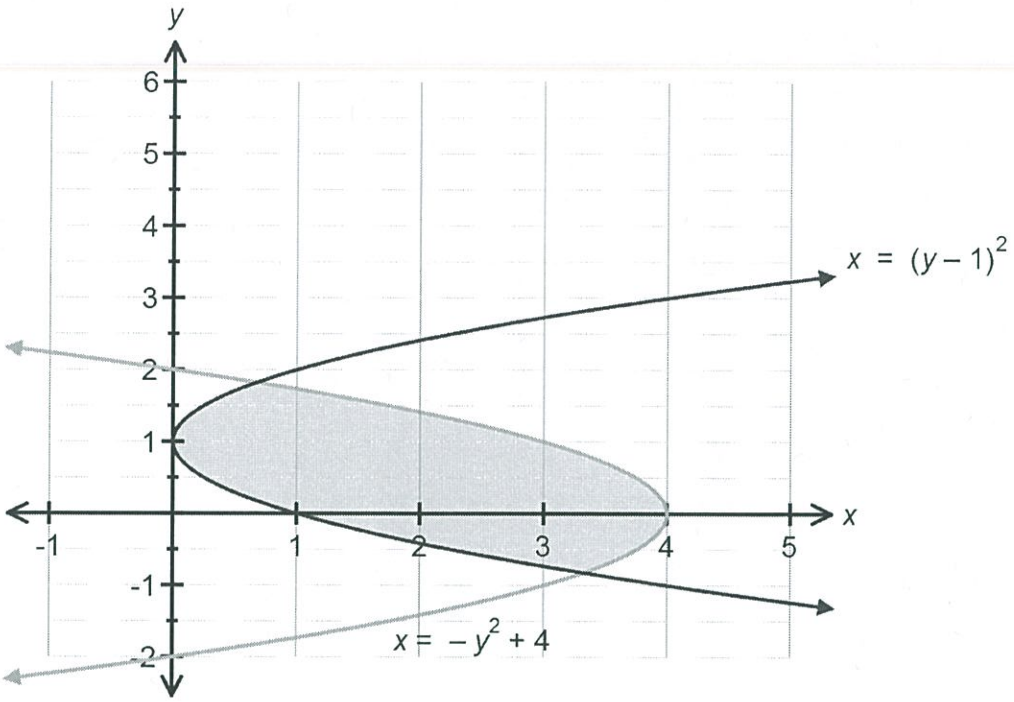
$A = 15000.$

(b) Hence determine the function that defines the value of her investment. (2 marks)

$$V(t) = 15\,000e^{0.06t}$$

Question 9 **(6 marks)**

Calculate the shaded area shown below, showing all relevant working.



for intersection:

solve$\left((y-1)^2 = -y^2+4, y\right)$

$y = 1.822288$ or $y = -0.822876$

$\therefore y \approx 1.82$ $\quad y \approx -0.83$

$x = 0.68$ $\quad x = 3.32.$

$$\text{Area} = \int_{-0.83}^{1.82} \left[(-y^2+4) - (y-1)^2\right] dy$$

$$= \left[-\frac{y^3}{3} + 4y - \frac{(y-1)^3}{3}\right]_{-0.83}^{1.83}$$

$= 6.17326$

$\approx \underline{6.173 \text{ units}^2}$

Question 10 **(4 marks)**

Show that $\int_1^2 \left(\frac{6x+4}{\sqrt{x}}\right) dx = 16\sqrt{2} - 12$.

*(Show sufficient work out please and use **exact** values)*

$$\int_1^2 \frac{6x+4}{\sqrt{x}} = \int_1^2 \left(6\sqrt{x} + \frac{4}{\sqrt{x}}\right) dx$$

$$= \int_1^2 \left(6x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}\right) dx$$

$$= \left(\frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4(x)^{\frac{1}{2}}}{\frac{1}{2}} \right)_1^2$$

$$= \left[\left(4x^{\frac{3}{2}} + 8\sqrt{x}\right)\right]_1^2 = \left(4\cdot 2^{\frac{3}{2}} + 8\sqrt{2}\right) - (4+8)$$

$$= 8\sqrt{2} + 8\sqrt{2} - 12$$

$$= 16\sqrt{2} - 12 = \text{RHS}$$

Question 11 **(3 marks)**

The area under the curve $f(x) = 4e^{kx}$ over the domain $0 \le x \le 10$ is $\frac{40}{3}\left(-e^{-3}+1\right)$.

Determine the value of k.

$$\therefore \int_0^{10} 4e^{kx}\, dx = \frac{40}{3}\left(-e^{-3}+1\right)$$

$$\left[\frac{4e^{kx}}{k}\right]_0^{10} = \frac{40}{3}\left(-e^{-3}+1\right)$$

$$\therefore \text{solve}\left(\frac{4e^{10k}}{k} - \frac{4}{k} = \frac{40}{3}\left(-e^{-3}+1\right), k\right)$$

$$\therefore k = -0{,}3$$

Question 12 **(4 marks)**

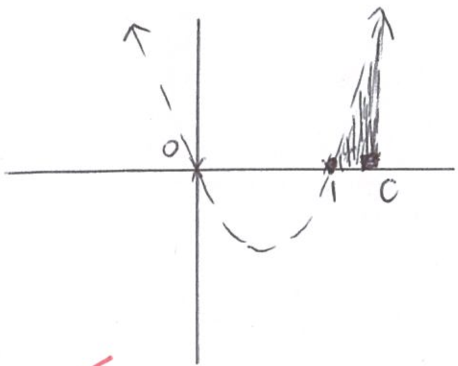
The area bound by the parabola $y = 6x^2 - 6x$, the x – axes and the lines $x = 1$ and $x = c, (c > 1)$ is equal to 1unit². Find the value of the constant.

x-intersections:

$6x^2 - 6x = 0.$

$6x(x-1) = 0$

$x = 0$ or $x = 1.$



$\therefore \int_1^c 6x^2 - 6x \, dx. = 1.$

$\therefore \left[\frac{6x^3}{3} - \frac{6x^2}{2}\right]_1^c = 1.$

$(2c^3 - 3c^2) - (2 - 3) = 1.$

$2c^3 - 3c^2 + 1 = 1.$

$2c^3 - 3c^2 = 0.$

$c^2(2c - 3) = 0.$

END OF PAPER 2

$c \neq 0$ (N.A. as $c > 1$) or $c = \frac{3}{2}.$